Measurement-based quantum computation

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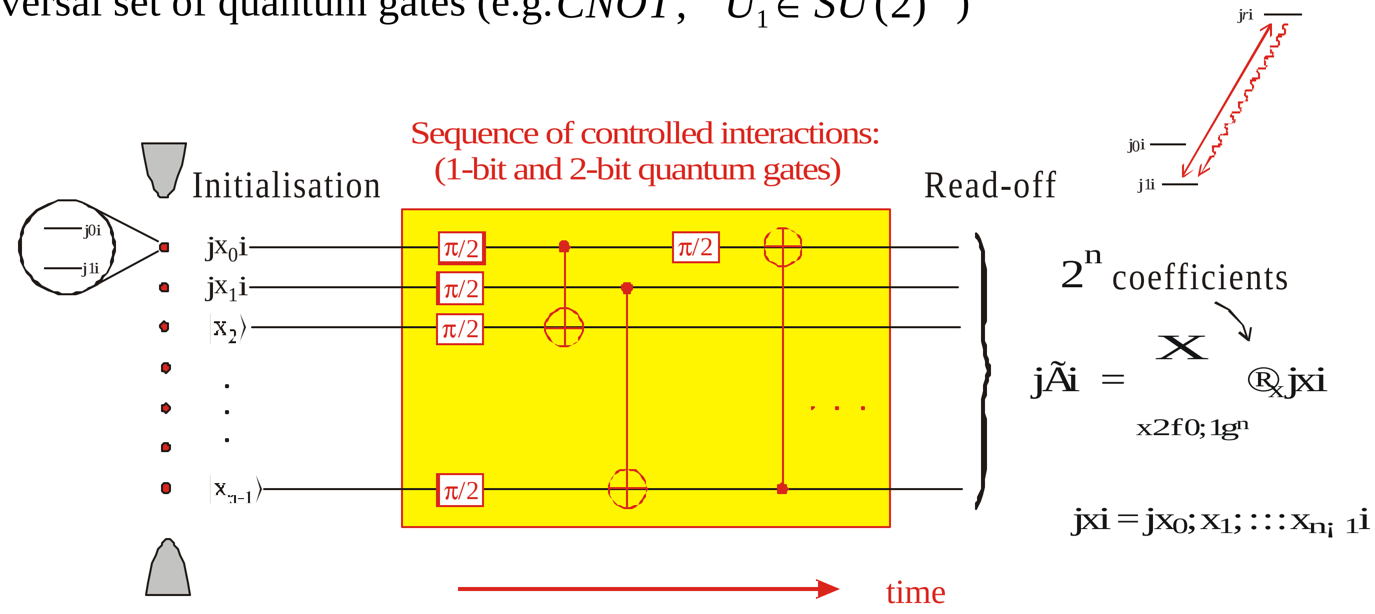
Plan of the talk

- Introduction
    - Quantum computational models
- One-way quantum computer (Summary)
    - Resources
    - Scheme
- Multi-particle entanglement
    - Cluster states
- Computational model of the $\mathrm{QC}_{\mathcal{C}}$
    - Universality proof
    - Complexity issues
- Implementation

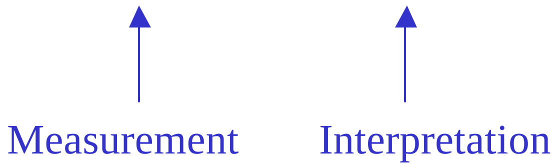
Quantum computational models

- Quantum Turing machine (Deutsch 86, Bernstein & Vazirani 93):
  - Generalization of classical universal Turing machine
  - Interference of computational paths in Hilbert space
- Quantum logical networks (Deutsch 89, Yao 93, Barenco et al. 95, ...):
  - Generalization of classical Boolean networks (reversible)
  - Universal set of quantum gates (e.g. $CNOT, \quad U_1 \in SU(2)$)



Quantum computational models II

- Measurement-based quantum computer (Raussendorf & Briegel 2000)
  - Information processing via sequence of 1-bit measurements on initialized (entangled) register („*quantum tape*“)
  - Can simulate any q.l. network efficiently
  - Principle of operation: Entanglement > Correlation > Logic



Measurement Interpretation

- Related work:
  - Teleportation as universal computational primitive (Gottesman & Chuang 1999)
  - 4-bit & 2-bit measurements as universal primitives (Nielsen 2001, Leung 2001)

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One-way quantum computer

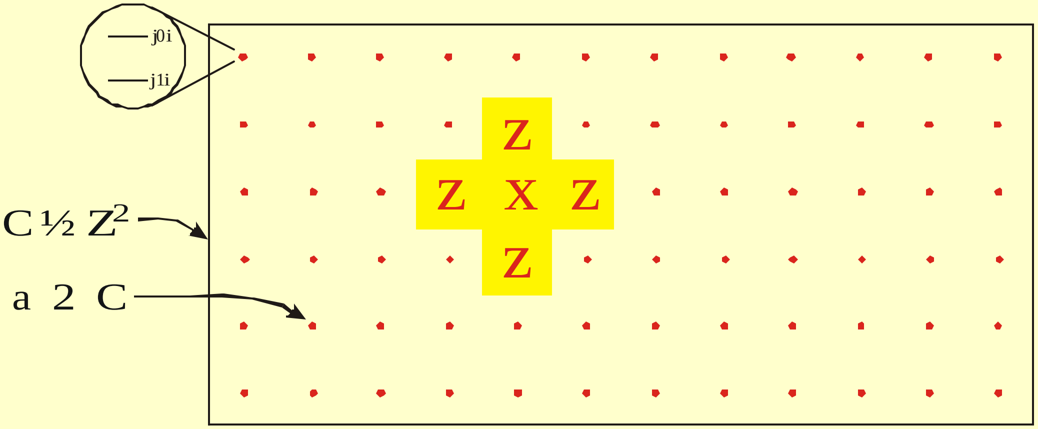
A: Resources

(1) Lattice of two-state systems in cluster state $jÁi_C$: $a\ 2\ C½Z^2$

$$K^{(a)} jÁi_C = jÁi_C$$

eigenvalue equations

$K^{(a)} ja\ 2\ C$ = group of correlation operators



(2) Projective von-Neumann measurements (1-bit)

$$P_{0;n} = j0i_n h0j$$
$$P_{1;n} = j1i_n h1j$$

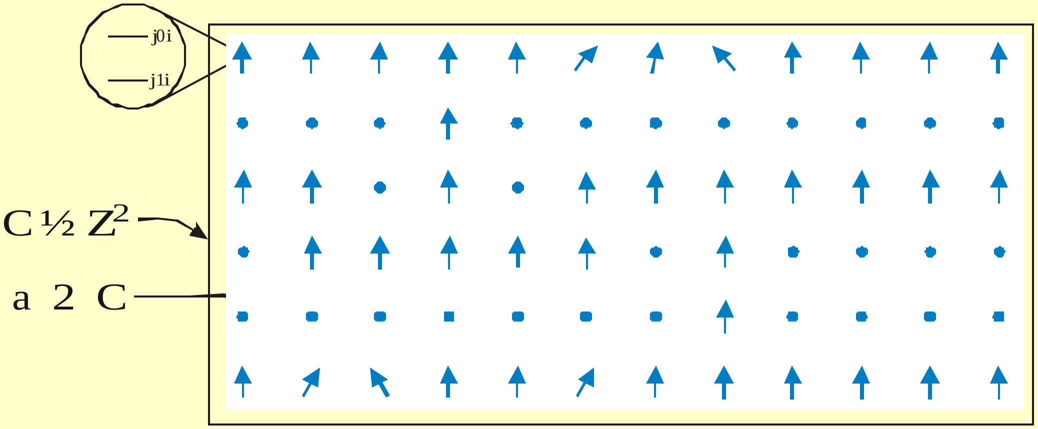
One-way quantum computer ...

B: Program

(1) Set of measurement directions

$n_{s}^{(a)}$ j a 2 C

fixed conditional



(2) Set of rules for processing the measurement results

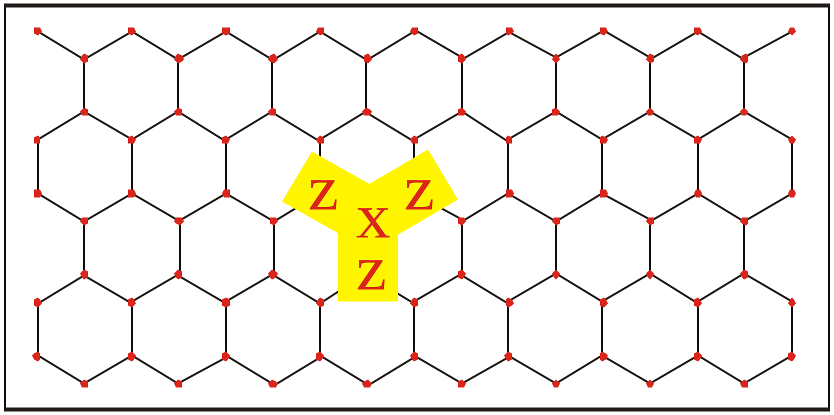
~ Transition function of QTM

Literature: Raussendorf & HJB
Phys. Rev. Lett. **86**, 910 (2001);
Phys. Rev. Lett. **86**, 5188 (2001).

Modifications

Cluster state on honeycomb lattice:

- Only 3 next-nearest neighbors
- Universal resource for q.c.
  via one-qubit measurements



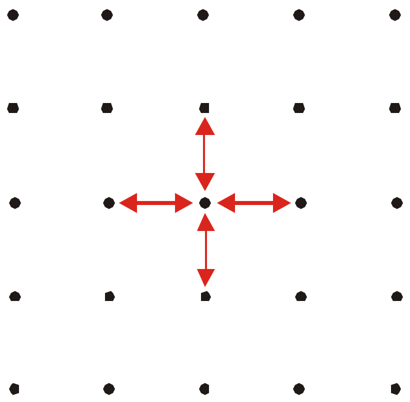
How do you create a cluster state?

$$K^{(a)}|\phi\rangle_C = \kappa_a|\phi\rangle_C \quad , \quad \kappa_a \in \{+1, -1\} \quad \forall a \in C$$

(1) Quantum memory in state $|\psi_0\rangle = \bigotimes_{a\in C} |0\rangle_{x,a}$ with $|0\rangle_x = \frac{1}{\sqrt{2}}(|0\rangle_z + |1\rangle_z)$

(qubits polarized in x direction)

(2) Switch on Ising interaction



$$H_{\text{int}} = -\frac{1}{4}\hbar g(t) \sum_{<a,a'>} \sigma_z^{(a)}\sigma_z^{(a')}$$

... for a certain period of time

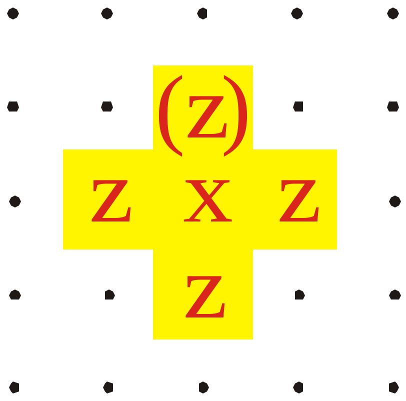
Alternatively:

(2´) Measure correlation (stabilizer) operators $K^{(a)}$ on $|\psi_0\rangle$:

$$\rightarrow \; |\phi\rangle_C =_{l.u.} \prod_{a\in C} \frac{1 \pm K^{(a)}}{2} |\psi_0\rangle$$

Corollary: 4-qubit measurements
sufficient for universal QC.

(Nielsen quant-ph/0108020)



Resources for QC

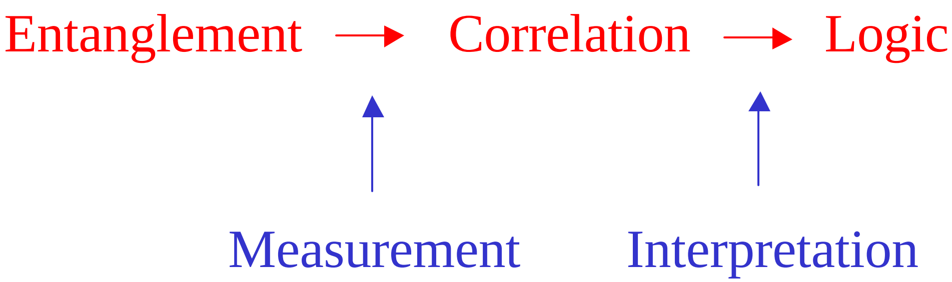
Measurement *versus* entanglement?

- Multi-qubit measurements of $K^{(a)}$ create entanglement between different qubits of the memory.

- 1-qubit measurements can only destroy entanglement.

Q: Interpretation of measurement
as a computational resource?

Conceptual framework of the QC_C



Entanglement → Correlation → Logic

Measurement

Interpretation

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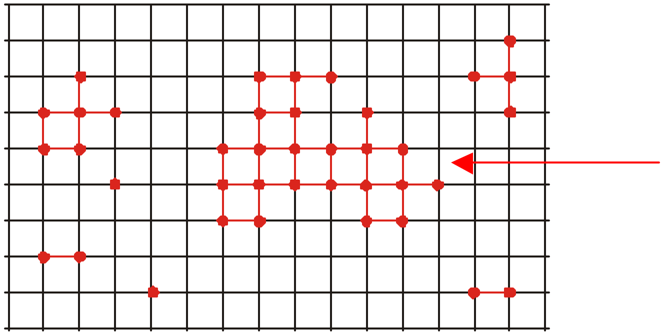
Cluster states II



● = qubit

cluster $C \subset \mathbb{Z}^d$

d=1,2,3

Hamiltonian

$$H_{\text{int}} = -\frac{1}{4}\hbar g(t) \sum_{\langle a;a'\rangle} \sigma_z^{(a)} \sigma_z^{(a')} \equiv_{\text{l.u.}} \hbar g(t) \sum_{\langle a;a'\rangle} \frac{1+\sigma_z^{(a)}}{2} \frac{1-\sigma_z^{(a')}}{2}$$

- Ising-type next neighbor interaction
- globally controllable interaction strength g(t)

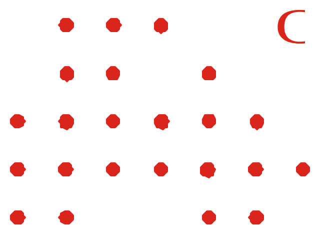
Unitary transform

$$S(\varphi) = \exp\left(-i\varphi \sum_{\langle a;a'\rangle} \frac{1+\sigma_z^{(a)}}{2} \frac{1-\sigma_z^{(a')}}{2}\right) \qquad \varphi = \int dt\, g(t)$$

- simultaneous conditional phase gate between all qubits

$$S(\varphi) = S(\varphi + 2\pi) \qquad \text{periodic}$$

Cluster states II ...



For initial state $\bigotimes_a |0\rangle_{x,a} \equiv \bigotimes_a \frac{1}{\sqrt{2}}\left(|0\rangle_a + |1\rangle_a\right)$, all qubits belonging to the same cluster *C* become entangled:

$$S(\varphi)\bigotimes_a |0\rangle_{x,a} = \begin{cases} |\phi\rangle_C; & \varphi = \pi, 3\pi, \ldots \quad \text{„maximum entanglement“} \\ \bigotimes_a |0\rangle_{x,a}; & \varphi = 2\pi, 4\pi, \ldots \quad \text{product state} \end{cases}$$

N.B.: Entanglement oscillations for $g(t)$ = const., $\varphi = gt$

Cluster state: $|\phi\rangle_C = S(\pi)\bigotimes_a \left(|0\rangle_a + |1\rangle_a\right)$

Cluster states II ...

D=1: $C = \{1, 2, \ldots, N\}$

$$|\phi\rangle_C \equiv |\phi_N\rangle = \prod_a \left(|0\rangle_a \sigma_z^{(a+1)} + |1\rangle_a \right), \qquad \sigma_z^{(N+1)} \equiv 1$$

multiply out!

$$= \left(|0\rangle_1 \sigma_z^{(2)} + |1\rangle_1 \right) \left(|0\rangle_2 \sigma_z^{(3)} + |1\rangle_2 \right) \left(|0\rangle_3 \sigma_z^{(4)} + |1\rangle_3 \right) \cdots \left(|0\rangle_N + |1\rangle_N \right)$$

Example:

$$|\phi_2\rangle = \left(|0\rangle_1 \sigma_z^{(2)} + |1\rangle_1 \right) \left(|0\rangle_2 + |1\rangle_2 \right)$$

$$= |0\rangle_1 \left(|0\rangle_2 - |1\rangle_2 \right) + |1\rangle_1 \left(|0\rangle_2 + |1\rangle_2 \right)$$

$$=_{\text{l.u.}} |0\rangle_1 |0\rangle_2 + |1\rangle_1 |1\rangle_2$$

Cluster states II ...

N=2: jÁ$_2$i =$_{\text{l.u.}}$ j0ij0i + j1ij1i

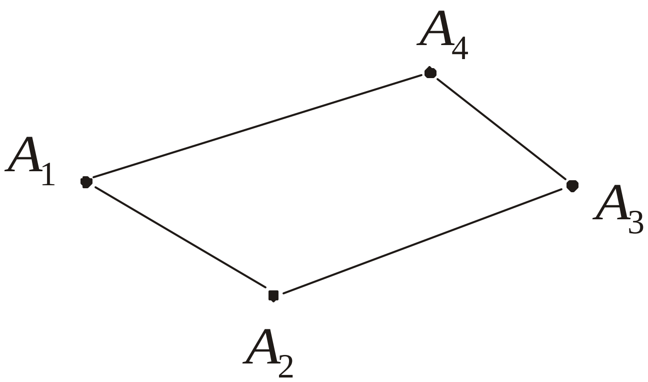
N=3: jÁ$_3$i =$_{\text{l.u.}}$ j0ij0ij0i + j1ij1ij1i

N=4: jÁ$_4$i =$_{\text{l.u.}}$ j0ij0ij0ij0i + j0ij0ij1ij1i + j1ij1ij0ij0i ¡ j1ij1ij1ij1i

compare: jGHZ$_4$i =$_{\text{l.u.}}$ j0ij0ij0ij0i + j1ij1ij1ij1i 6=$_{\text{l.u.}}$ jÁ$_4$i

Q: How can we distinguish jÁ$_4$i and jGHZ$_4$i
in operational terms?

ϕ_4 *versus* GHZ$_4$



Similarity: Any two qubits can be projected into a Bell state by local measurements on the other qubits (& class. communication).

Difference: It is impossible to destroy all entanglement of $|\phi_4\rangle$ by a single local operation, such as von Neumann measurement or tracing.

Schmidt measure of entanglement:

Partitioning	$\|GHZ_4\rangle$	$\|\phi_4\rangle$
$(A_1)(A_2)(A_3)(A_4)$	1	2
$(A_1A_2)(A_3)(A_4)$	1	1
$(A_1A_2)(A_3A_4)$	1	1
$(A_1A_3)(A_2A_4)$	1	2
$(A_1A_2A_3)A_4$	1	1

Eisert & HJB, Phys. Rev. A **64** 022306 (2001)

$$|GHZ_4\rangle =_{\text{l.u.}} |0\rangle|0\rangle|0\rangle|0\rangle + |1\rangle|1\rangle|1\rangle|1\rangle$$

$$|\phi_4\rangle =_{\text{l.u.}} |0\rangle|0\rangle|0\rangle|0\rangle + |0\rangle|0\rangle|1\rangle|1\rangle + |1\rangle|1\rangle|0\rangle|0\rangle - |1\rangle|1\rangle|1\rangle|1\rangle$$

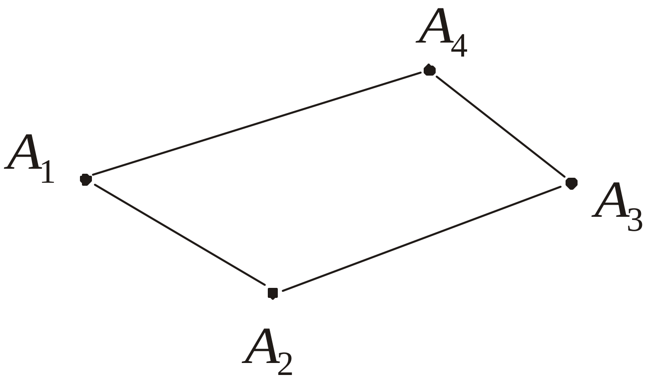
Maximal connectedness & Persistency of entanglement

HJB & Raussendorf, Phys. Rev. Lett. **86,** 9910 (2001)

Definition 1: (Max. connectedness) The quantum mechanical state of a set $C = \{1,2,...n\}$ of n qubits is maximally connected if any two qubits of the set C can be projected, with certainty, into a pure Bell state by local measurements on a subset of the other qubits.

Definition 2: (Persistency) The persistency of entanglement P_e of an entangled state of n qubits is the minimum number of local measurements such that, for all measurement outcomes, the state is completely disentangled (product state).

$$P_e(|GHZ_4\rangle) = 1$$
$$P_e(|\phi_4\rangle) = 2$$



Entanglement properties of the cluster state

(1) $|\phi_N\rangle$ is *maximally connected*.

(2) Persistency of entanglement $P_e(|\phi_N\rangle) = \lfloor \frac{N}{2} \rfloor$

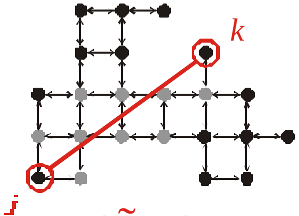
(3) Expansion of $|\phi_N\rangle$ into product basis contains *exponentially many terms*:

$$|\phi_N\rangle = \sum_{j=1}^{R} \alpha_j |\phi_1^{(j)}\rangle |\phi_2^{(j)}\rangle \cdots |\phi_N^{(j)}\rangle \quad , \qquad R \geq r = 2^{\lfloor \frac{N}{2} \rfloor}$$

Schmidt measure: $P_S(|\phi_N\rangle) = \log_2(r) = \lfloor \frac{N}{2} \rfloor = P_e(|\phi_N\rangle) =$ persistency

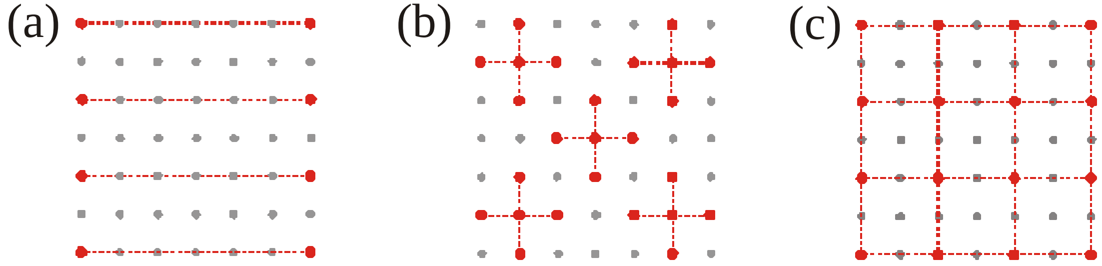
N.B.: Similar results hold for 2D and 3D

Entanglement resource



2D & 3D clusters are maximally connected and represent an entanglement resource

- Project out Bell and GHZ states by 1-particle measurements:



4 Bell states   5 GHZ states   1 GHZ state

[Grey ● symbolize σ_z measurements (between the ┈┈┈ lines)
and σ_x measurenents (on the ┈┈┈ lines)]

- Project out any *n*-particle entangled state (*!*) by 1-particle measurements on *m* >> *n* particles.

  Quantum state engineering via ``lithographic measurements´´

HJB & Raussendorf, Phys. Rev. Lett. **86,** 9910 (2001)

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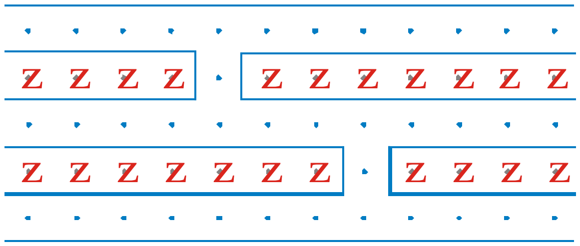
Quantum computing on a cluster

Structuring the cluster: $[\sigma_z^{(a)}, S(\varphi)] = 0;\ \forall a \in C$

z-measurements commute with entanglement operation

$$H_{int} \sim \sigma_z^{(a)}\sigma_z^{(a')}$$

measured qubits set $M \subset C$



unmeasured qubits set $N = C \setminus M$

$$\prod_{a \in M} \frac{1+\sigma_z^{(a)}}{2}\,|\psi\rangle_C =_{l.u.} |\psi\rangle_{C\setminus M} = |\psi\rangle_N$$

N = carrier of quantum logic network

→ reduced cluster state: $K^{(a)}|\psi\rangle_N = \pm|\psi\rangle_N;\ \forall a \in N$

$$K^{(a)}|\psi\rangle_N = \pm|\psi\rangle_N$$

Quantum computing on a cluster

Entanglement ⟶ Correlation:

Observe $\langle\psi|\mathbf{n}\cdot\boldsymbol{\sigma}^{(a)}|\psi\rangle = 0\;;\;\forall n,a$ while $\langle\psi|K^{(a)}|\psi\rangle = \pm 1\;;\;\forall a$

$$K^{(a)} = \sigma_x^{(a)} \bigotimes_{<a,a'>} \sigma_z^{(a')} \in \left\{ ZXZ\;,\; \begin{smallmatrix} & Z & \\ Z & X & Z \end{smallmatrix}\;,\; \begin{smallmatrix} Z & X & Z \\ & Z & \end{smallmatrix} \right\}$$

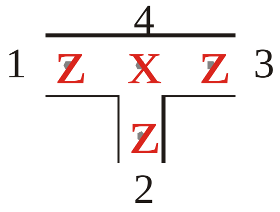
Possible outcomes of potential σ_z and σ_x measurements:

$$z_a\,;\,x_a \in \{+1;-1\}$$

random variables

... obey strict correlations $x_a \prod_{<a;a'>} z_{a'} = \pm 1$

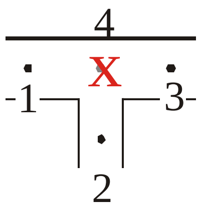
Example:



$$x_4 z_1 z_2 z_3 = 1$$

Quantum computing on a cluster

Measure only qubit 4:



suppose $x_4 = +1 \longrightarrow$

$z_3 = z_1 z_2$

conditional correlations

Write $z = (-1)^{\zeta}$ with logical variable $\zeta \in \{0, 1\}$

$$\zeta_3 = \zeta_1 + \zeta_2 \bmod 2 = \mathrm{XOR}(\zeta_1, \zeta_2)$$

similarly for $x_4 = -1$: $\zeta_3 = 1 + \zeta_1 + \zeta_2 \bmod 2 = \mathrm{NOT} \circ \mathrm{XOR}(\zeta_1, \zeta_2)$

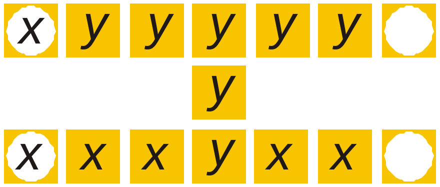
N.B.: Logical relation for *potential z measurements* on qubits 1,2,3.

Quantum logic gates

Controlled NOT:

$$\mathrm{CNOT}(c,t) = |0\rangle_c\langle 0| \otimes 1^{(t)} + |1\rangle_c\langle 1| \otimes \sigma_x^{(t)}$$

Measure:



Obtain:

$$U' = \sigma_z^{(c)\,\gamma_c}\,\sigma_x^{(c)\,\lambda_c}\,\sigma_z^{(t)\,\gamma_t}\,\sigma_x^{(t)\,\lambda_t}\;\mathrm{CNOT}$$

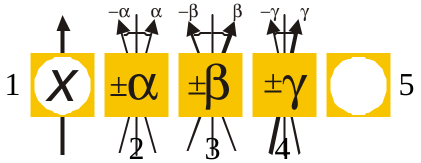
byproduct operators

$\gamma_c, \lambda_c, \gamma_t, \lambda_t \in \{0, 1\}$ depend on the random measurement results

Quantum logic gates

Rotation: $U = U_x(®)U_z(¯)U_x(°)$

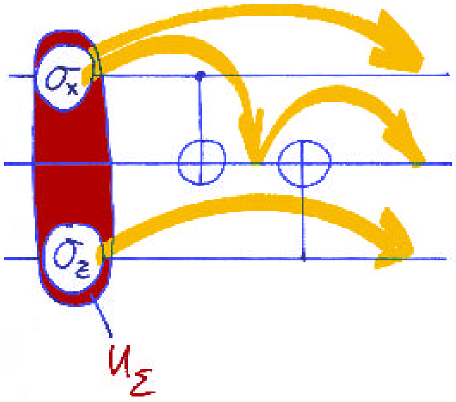
Measure:



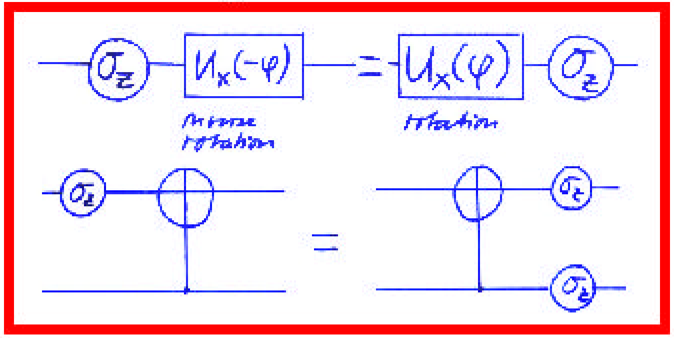
Obtain: $U^o = ¾_x^{(s_1+s_2)} ¾_z^{(s_1+s_3)}\,U$

- Temporal ordering of measurements!

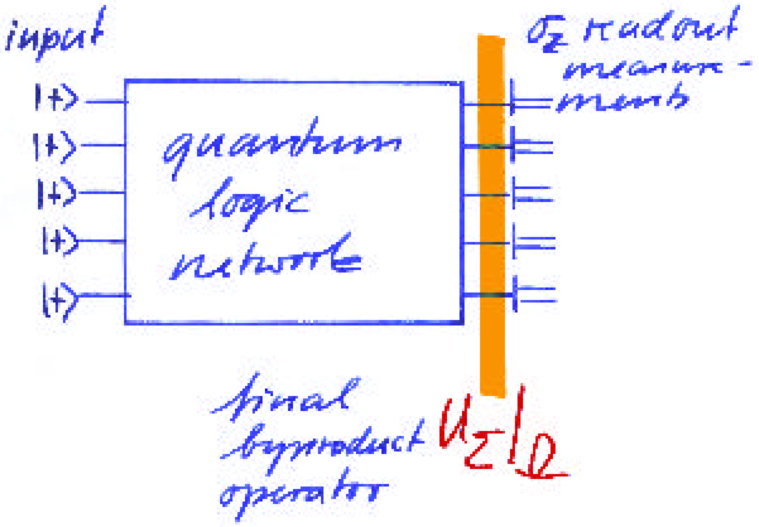
Byproduct operator propagation



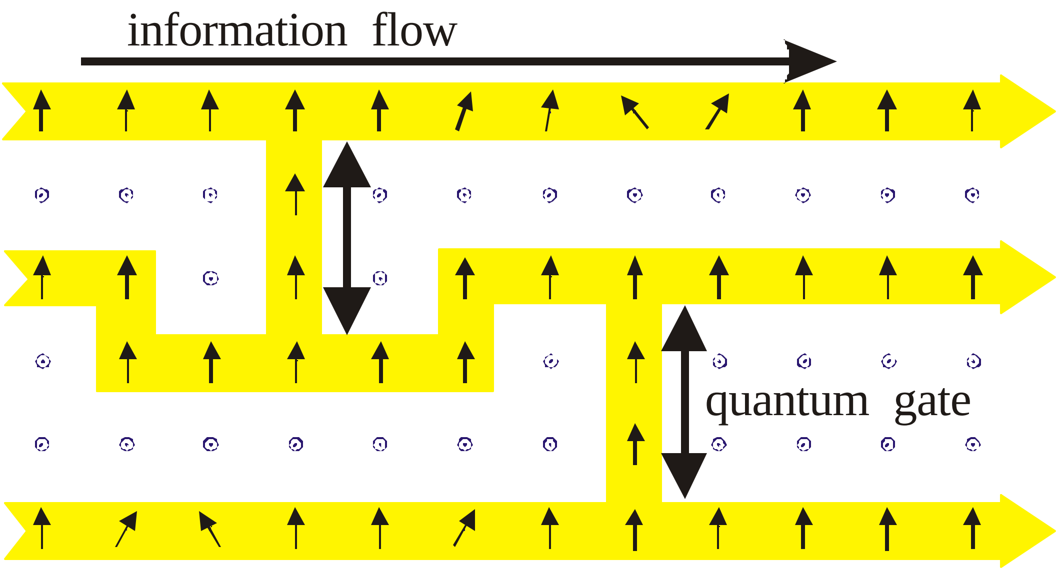
propagation relations



accumulated byproduct operator $U_S|_W$



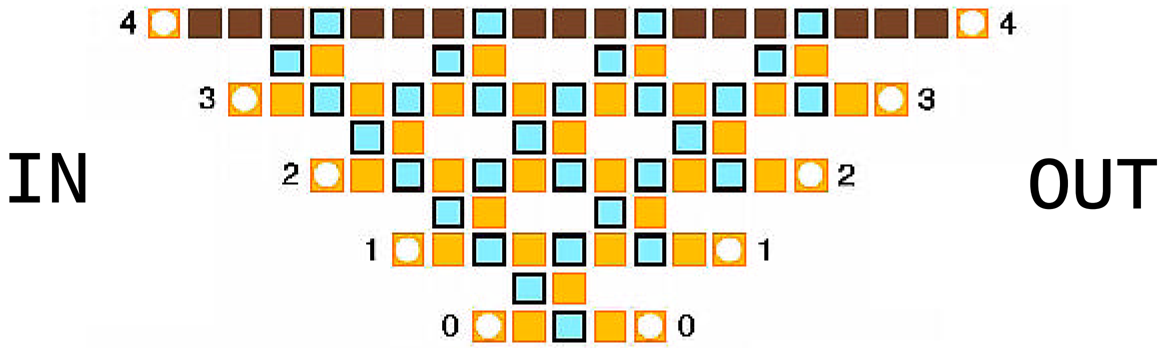
Simulation of logical networks (circuits)



measurements:
⊙ in Z direction
↑ in X direction
↖ in X-Y plane

Raussendorf & HJB, Phys. Rev. Lett. **86,** 5188 (2001)

Quantum Fourier Transform



🟧 x measurement

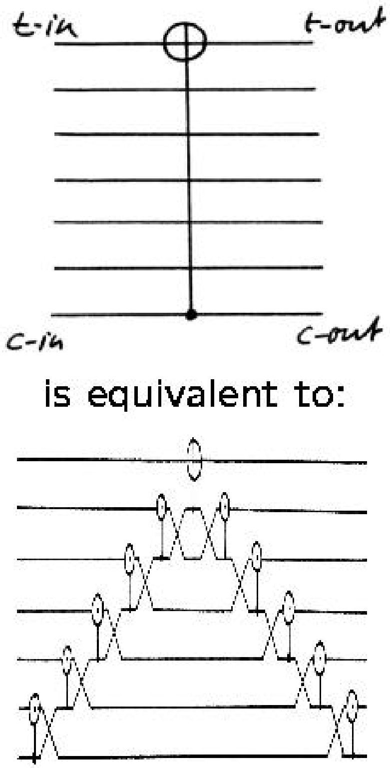
🟫 y measurement

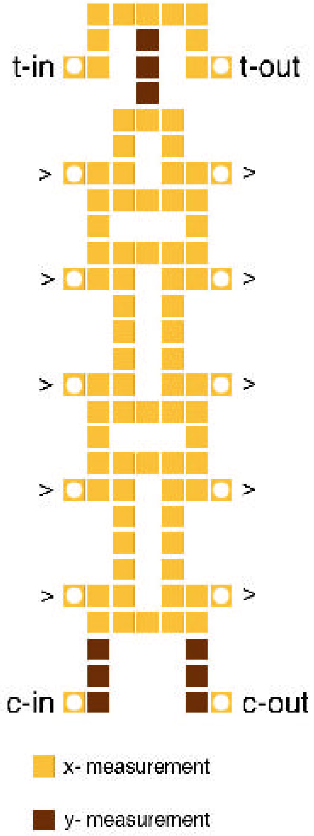
▣ measurement in x-y plane

Browne, Raussendorf, HJB, unpublished (2001)

General CNOT gate

`CNOT` between arbitrary qubits ...

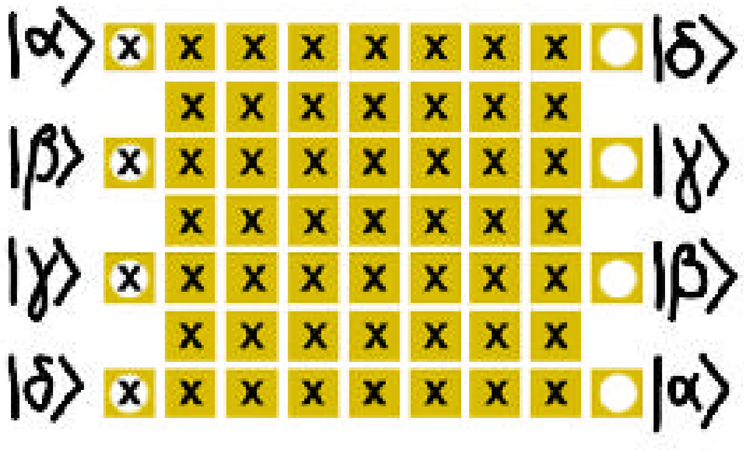




... requires slice of constant thickness (6 qubits)

Multi-qubit swap gate

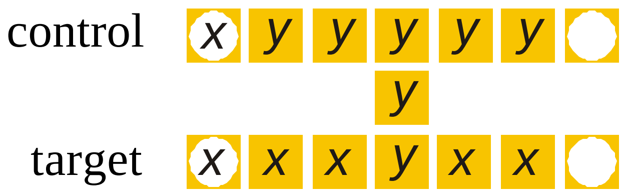
IN



OUT

Beyond the network picture

- All measurements that realize CNOT, Hadamard and $\pi/2$ phase gates can be performed simultaneously



CNOT gate

Hadamard gate

$\pi/2$ phase gate

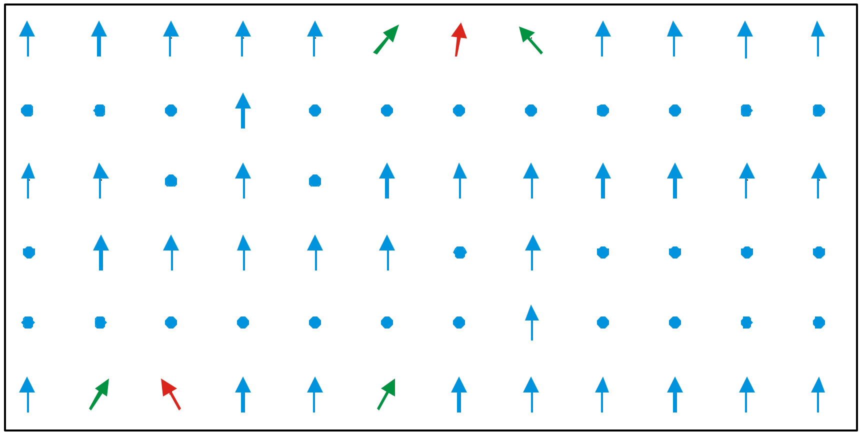
Reason: Only σ_x and σ_y measurements involved. Choice of bases does not depend on results of other measurements.

→ Unit logical depth for circuits in the Clifford group

Raussendorf & HJB, quant-ph/0108067

Sets M_t of simultaneously measurable qubits

- Measure every qubit at the earliest possible time.



- Cluster C decomposed into subsets $M_j \subset C$

$$C = \bigcup_{j \in I} M_j \;, \quad M_j \cap M_k = \emptyset \;; \quad j \neq k$$

  - All qubits of the same subset may be measured at the same time
  - Temporal order between different subsets

- Computation $\hat{=}$ Series of simultaneous 1-qubit measurements on disjoint sets $M_0 \,;\, M_1 \,;\, M_2 \,;\, \ldots$ of qubits.

Raussendorf & HJB, quant-ph/0108067

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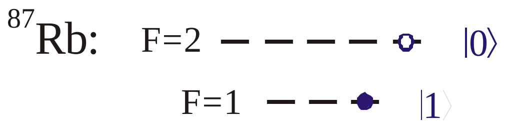
Implementations

- Implementation of controlled Ising interaction has been proposed for various systems:

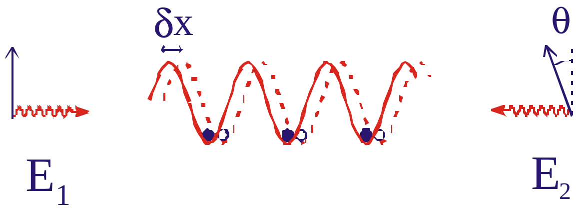
– Optical lattices	(Jaksch et al., PRL 82, 1999)
– Magnetic microtraps	(Calarco et al., PRA 61, 2000)
– Josephson loops	(Mooij et al., Science 285, 1999)
– Arrays of quantum dots (?)	(Tanamoto, quant-ph/0009030)

Implementation in optical lattices

Rubidium:



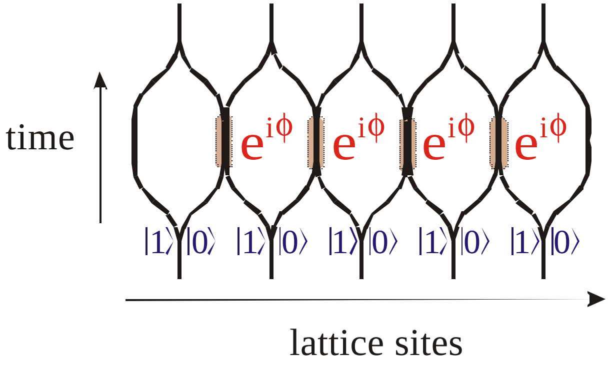
Optical lattices:
("lin-angle-lin" config.)



Multi-atom interferometer:

$$\phi = \frac{1}{\hbar}\int dt\, \Delta E(t)$$

interaction phase
(ultracold collisions*)



*Jaksch et al., *Phys. Rev. Lett.* **82**, 1975 (1999)

Corresponds formally to controlled Ising interaction